iTalk2Learn
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Deliverable 1.1

State-of-the-art for intelligent tutoring systems and exploratory learning environments

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(with contributions from all partners involved in WP1 and review from Whizz, BBK)

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0.9	14/10/2013	IOE and RUB version addressing BBK and Whizz comments
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Executive Summary

This deliverable presents a key building block in the iTalk2Learn project by preparing the ground for the domain-specific and pedagogical aspects that facilitate the development of the exploratory learning environment and its tasks as well as the intervention models for the learning platform. The emphasis is on definitions, to provide a common ground for partners when preparing for the work around adaptive intelligence (WP2), intuitive interaction interfaces (WP3), planning evaluation activities (WP5) and to a lesser extent deployment and integration of the learning platform (WP4). After a general introduction (Section 1), Section 2.1 defines procedural and conceptual knowledge - the two components students require for robust mathematical learning. It explains how, through the lenses of a cognitive science and a mathematics education perspective, procedural knowledge (knowledge about and application of procedures) and conceptual knowledge (implicit or explicit understanding about underlying principles and structures of a domain) develop iteratively through different pedagogical approaches to create robust mathematical knowledge.

The deliverable also outlines the consortium's decision to focus on fractions that are widely accepted as being a very difficult aspect of mathematics to teach and learn. Section 2.2 highlights how the five interpretations of fractions (the sub-constructs - part-whole, ratio, operator, quotient and measure - defined in Section 2.2.1) are a significant reason for this. In addition to this, there is a range of representations (area/region, number line, set of objects, liquid measures and symbol) that can be used by teachers and students to represent fractions and these are also discussed. Although these interpretations and representations are available to teachers, it has been shown worldwide that receiving limited exposure to these has an impact in students' understanding. The implication for iTalk2Learn is to ensure students have access to a wide range of interpretations and representations to address this current imbalance.

Sections 3 identifies how cognitive, domain-specific and pedagogic approaches can be reflected in Intelligent Tutoring Systems (ITSs) and Exploratory Learning Environments (ELEs). The consortium will use tasks in Whizz and Fractions Tutor to support students' procedural knowledge and ELE tasks to support students' conceptual knowledge of fractions and particularly addition and subtraction of fractions. Section 3 discusses the affordances offered by ITSs, which include facilitated "drill-and-practice" of routine problems with hints and feedback in support of students' procedural knowledge. Section 4 details how ELEs offer students the opportunity to experience tools and tasks that are built on an underlying, conceptually-based structure. We hypothesize that a carefully constructed combination of ITSs and ELEs will lead to robust mathematical learning. Section 4 highlights the need for intelligent support while students undertake exploratory tasks. This raises implications with respect to the design of the ELE in iTalk2Learn in that it needs to provide access to sufficient unambiguous information (in order to enable inference based on students' interactions), but not be intrusive and make full use of students' actions in the user interface (neither interfering too much nor limiting the exploratory nature of the ELE).

Section 5 provides a summary and the conclusions emerging from the deliverable, drawing out the implications for the iTalk2Learn project, including how procedural knowledge is developed through ITSs and conceptual knowledge is developed through ELEs and the tasks set within them.

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List of Abbreviations

UHi	Universitat Hildesheim
IOE	Institute of Education, University of London
TL	Testaluna SRL
RUB	Ruhr-Universität Bochum
BBK	Birkbeck College – University of London
Whizz	WHIZZ Education Limited
SAIL	SAIL Labs Technology AG
ELE	Exploratory Learning Environment
MW	Microworlds
ITS	Intelligent Tutoring System
WP	Work package

1. General Introduction

The objective of this deliverable is mainly to review the state-of-the-art of the two types of educational technology related to the iTalk2Learn project: intelligent tutoring systems (ITSs) and exploratory learning environments (ELEs) and their relationship to procedural and conceptual learning. In reference to the iTalk2Learn objectives, WP1 aims to provide the pedagogical background required in the project with respect to learning processes and possible guidance or support required in elementary fractions, the domain chosen by the project. As mentioned in other deliverables, the project selected fractions, and addition and subtraction in particular, as the target domain because of the widely acknowledged difficulty that students have in learning fractions and the richness fractions afford with respect to different representations and interpretations. The WP1 work contributes indirectly to all the objectives of the project but mostly to objective 3 and 4. We repeat the objectives below and the WP structure of the project for completeness.

The iTalk2Learn objectives:

1. Provide an open-source platform for intelligent support systems integrating structured practice and exploratory, conceptually-oriented learning
2. Provide state-of-the-art and highly innovative reference implementations of plugins for the platform that could be used in a wide range of application domains
3. Promote our understanding of the role of the different modalities of speech and direct manipulation of multiple or alternative representations in learning elementary mathematics through digital technologies
4. A summative evaluation of activities and support features generated by our intelligent learning support platform

The iTalk2Learn work packages:

WP number	WP name	Lead beneficiary
1	Robust Learning in Elementary Mathematics	IOE
2	Adaptive Intelligence for Robust Learning Support	UHi
3	Intuitive Interaction Interfaces for Elementary Mathematics	TL/SAIL
4	Deployment and Integration	BBK
5	Data Collection and Evaluation	RUB
6	Dissemination and Exploitation	Whizz
7	Project Management	UHi

As this is the first deliverable in WP1, Section 2 provides background knowledge with respect to procedural and conceptual knowledge and sets the scene for the WP1 work related to interpretations and representations of fractions in mathematics education. The emphasis here is on definition, to provide a common ground for partners when preparing for the work around adaptive intelligence (WP2), intuitive interaction interfaces (WP3), planning evaluation activities (WP5) and to a lesser extent deployment and integration of the learning platform (WP4).

The rest of the deliverable puts emphasis on the review of the state-of-the-art and implications for the project. As a complete review was not possible and unnecessary in the scope of the project, the consortium focused mostly on either very relevant or recent work (last 5 years) and identified useful axes of analysis that helped the project move forward (particularly since chronologically this deliverable reports on the first task in WP1). In particular throughout the WP1 work we focused primarily on intelligent and exploratory learning environments for students in our wider target group and aimed to complement D3.2 (that looks at the interactive features of ELEs) and D2.1 (that discusses the more technical aspects of both ITSs and ELEs). In more detail the concentrated review efforts of the state-of-the-art across the partners focuses on:

(a) the intelligent and adaptive features of ITSs particularly with respect to procedural knowledge (this is partially reported here and in D2.1)

(b) the kinds of interaction and affordances of the exploratory environments (reported here and in D3.2)

(c) the types of representations employed and required for elementary mathematics (this is partially reported here in the background section but it is ongoing work through T1.2 and will reported in D1.2)

The overarching question for this deliverable was: How do young children construct mathematical knowledge in these environments?

The review therefore has provided the consortium (both through the availability of the deliverables and cross-partners presentations during project meetings) a good grasp of the state-of-the-art and helped not only cross-feed across WPs but also to identify fruitful research avenues (keeping also an eye to possible exploitation avenues mostly for the benefit of the industrial partners as well as the future impact of the project).

Section 3 reviews relevant intelligent tutoring systems, with a focus mostly on the support provided within the two ITSs that we will utilise in the project (Whizz and Fractions Tutor). Section 4 reviews ELEs and the need and possibility for intelligent support. Section 5 provides a summary and implications for the iTalk2Learn project.

2. Background & Terminology

At an early stage of the project we identified and agreed on key terminology that will be used by partners. In order to answer the overarching question, 'How do young children construct robust mathematical knowledge when interacting with intelligent systems or exploratory learning environments?' mathematical knowledge is defined as procedural and conceptual knowledge. The partners' agreed interpretations of procedural and conceptual knowledge are discussed in Section 2.1. Section 2.2 focuses on domain-specific terminology related to the interpretation and representation of fractions, that is necessary both for WP2 and WP3.

2.1 Procedural and conceptual knowledge

In order to design and develop an effective learning platform for the iTalk2Learn project we needed to first define, within the parameters of this project (both from cognitive science and mathematics education perspectives), the broad types of procedural and conceptual knowledge that students construct. We begin by defining procedural and conceptual knowledge and then discuss the interaction between the two types of knowledge. Identifying how intelligent tutoring systems (ITS) and exploratory learning environments (ELEs) support procedural and conceptual knowledge is necessary in the project to ensure students are presented with the most appropriate task at any given time during their interaction with the learning platform.

Research on cognitive psychology and in mathematics education shows that robust knowledge about a certain domain (as here about fractions) consists of two types of knowledge, namely procedural and conceptual knowledge (Anderson, 1987; Rittle-Johnson, Siegler, & Alibali, 2001; Skemp, 1976).

2.1.1 Defining procedural knowledge

Procedural knowledge is defined as knowledge about and application of procedures (Rittle-Johnson & Alibali, 1999). Procedures are an action sequence of e.g. mathematical problem-solving steps (Rittle-Johnson & Alibali, 1999). The main aspect of procedural knowledge is in knowing how to apply a rule in order to solve a problem. According to Anderson's ACT-R model (Anderson, 1982, 1983, 1987; Anderson & Lebiere, 1998) procedural knowledge becomes implicit with increasing practice. Skemp (1976) identified three advantages of procedural knowledge: it can be easier to understand, the answers can be found more quickly and reliably, and students feel a sense of success when their answers are right.

2.1.2 Defining conceptual knowledge

On the other hand, conceptual knowledge is defined as implicit or explicit understanding about underlying principles and structures of a domain (Rittle-Johnson & Alibali, 1999). The focus of this type of knowledge lies on understanding why, for example, different mathematical principles refer to each other and on making sense of these connections. Skemp (1976) outlined four advantages of conceptual knowledge: it is more adaptable to new tasks, it is easier to remember, it is motivational to learn, and it is organic so new material can be learned relationally.

2.1.3 The interaction between procedural and conceptual knowledge

Both types of knowledge develop over the same period of time (Canobi, Reeve, & Pattison, 2003; Fuson, 1988; LeFevre et al., 2006) and evolve in a relationship of mutual dependence (Rittle-Johnson & Koedinger, 2009). Conceptual and procedural knowledge develop iteratively “with increases in one type of knowledge leading to gains in the other type of knowledge, which trigger new increases in the first” (Rittle-Johnson, et al., 2001). Against this background there is some debate about the finer-grained interaction, the focus of interest concerns the question, whether conceptual or procedural knowledge has a greater impact on the mutual development (Rittle-Johnson & Koedinger, 2009). Some studies from the field of cognitive psychology (e.g. Byrnes & Wasik, 1991) show that conceptual knowledge might have a greater impact on the development of procedural knowledge. In mathematics education (and depending on the subject matter) some argue that procedural understanding precedes conceptual understanding (Gray & Tall, 1994; Sfard, 1991). Of course, the findings are domain-specific and in a review of the literature on this issue, (Rittle-Johnson & Siegler, 1998) emphasize that there is no sufficient evidence to support a fixed order over another with respect to the acquisition of procedures or concepts.

2.1.4 Developing procedural and conceptual knowledge

In this context, the question about how conceptual knowledge in particular, and procedural knowledge develop, comes into play: In line with the Knowledge-Learning-Instruction Framework (Koedinger, Corbett, & Perfetti, 2012) focusing on the relationship between different knowledge components, learning processes and instructional approaches, procedural and conceptual knowledge evolve differently: While procedural knowledge is acquired through repeated (structured) practice and deepening of problem-solving procedures (Anderson, Boyle, Corbett, & Lewis, 1990), conceptual knowledge develops by providing students with exploratory learning activities and encouraging reflection and self-explanation (Ainsworth & Loizou, 2003; Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Lewis, 1988; VanLehn, 1999). In doing so, students are enabled to abstract concrete information, construct schemata (Koedinger, et al., 2012) and hence develop conceptual knowledge. In order to support the structured practice activities most effectively, students need to be guided through the problem-solving process step-by-step and receive feedback. In the case of the exploratory learning activities students need to be provided with a space for discovering the underlying (mathematical) principles. As we discuss in Section 4 this discovery also requires significant pedagogic support. As we aim at the design of an effective learning environment with our iTalk2Learn platform, we need to take into account the specificity of these two different approaches of knowledge development and how they are reflected in ITSs and ELEs. This is explored in detail in Sections 3 and 4 below. We discuss our considerations and implications for iTalk2Lean in Section 5.

2.2 Interpretations and representations of fractions

A review of the related work in mathematics learning is out of the scope of this deliverable (more will be reported in D1.2 and D1.3), but for the purposes of completeness and ease of understanding (and because they have been fundamental to consortium discussions as of M12), we refer briefly to representations and interpretations of fractions here. As mentioned in the Introduction, we aim at helping students develop robust knowledge in the field of fractions in general, and addition and subtraction in particular, mostly because it is widely accepted that fractions is one of the most difficult aspect of mathematics to teach and learn (Charalambous & Pitta-Pantazi, 2007). The difficulty arises because of the complexity of fractions, due to the number of ways they can be interpreted (see Section 2.2.1 for explanation of these). In addition to this, teachers have at their disposal a number of representations that they can draw upon to teach fractions (see Section 2.2.2). Using a range of representations, while potentially supporting students' cognitive development, can also add to the complexity of fractions.

The purpose of including definitions of interpretations and representations of fractions in this deliverable is to raise the issue of complexity and to present the shared understanding that the project partners have developed as of M12. The relationship between interpretations and representations and how the design of the ELE and exploratory tasks utilises these findings will be reported in D1.2. How student difficulties arising from interpretations and representations and the possible interventions in the learning platform to support students will be reported in D1.3.

2.2.1 Interpretations of fractions

In order to enable students to develop their knowledge about fractions we need to take into account that fractions learning implies appreciating the several interpretations (also referred to as subconstructs) of fractions (Kieran, 1976, 1993). These have inevitable overlaps, and this is one reason why fractions cause such difficulty. These interpretations are part-whole, ratio, operator, quotient, and measure. We use the fraction 3/4 to exemplify each interpretation in Table 1.

Table 1. Interpretations of fractions, exemplified using 3/4.

Interpretation	Commentary
Part-whole	In part-whole cases, a continuous quantity or a set of discrete objects is partitioned into a number of equal-sized parts. In this interpretation, the numerator must be smaller than the denominator. E.g. a pie is divided into four equal parts (quarters) and three are eaten, so 3/4 of the pie has been eaten. Students need to understand that (a) the parts into which the whole is partitioned must be of equal size; (b) the parts, taken together, must equal to the whole; (c) the more parts the whole is divided into, the smaller the parts become; and (d) the relationship between the parts and the whole is conserved, regardless of the size, shape or orientation of the equivalent parts (Leung, 2009). Part-whole is the most common interpretation used in elementary school exercise books (Alajmi, 2012). It is the interpretation that students perform consistently higher in compared to the other interpretations (Charalambous & Pitta-Pantazi, 2007; Hannula, 2003; Ni, 2001).

Ratio	A fraction can be seen as a ratio of two quantities; in this case it is seen as a part-part interpretation and it is considered to be a comparative index rather than a number (Carraher, 1996). E.g. Three parts out of every four are red. For this interpretation, students need to understand the relative nature of the quantities. They also need to know that when two quantities in the ratio are multiplied by the same positive number, the value of the ratio is unchanged (Leung, 2009).
Operator	A fraction is interpreted as an operator when it is applied as a function to a number, set or object. E.g. showing 3/4 of a pie chart or finding 3/4 of 24. In the operator interpretation, students need to understand that it is possible to interpret a fractional multiplier in a variety of ways; name a single fraction to describe a composite operation, when two multiplicative operations are performed; and relate outputs to inputs (Leung, 2009).
Quotient	The quotient interpretation is the result of a division. It results in a number that can be placed on a number line. E.g. 3 ÷ 4 = 3/4. For this interpretation, students should be able to identify fractions with division and understand the role of the dividend and the divisor in this operation (Leung, 2009).
Measure	In a measure interpretation a unit fraction is identified (e.g. 1/4) and how many units are used repeatedly determine a distance from a predetermined starting point (e.g. 3 x 1/4 = 3/4). In this interpretation, students should be able to use given unit interval to measure any distance from the origin; locate a number on a number line; and identify a number represented by a point on the number line (Leung, 2009) or region (Pantziara & Philippou, 2012).

It is the focus of the part-whole interpretation in curricula around the world that has led many researchers to focus on other interpretations and to question the extent to which part-whole interpretations impact on children's understanding of fractions (Behr, Lesh, Post, & Silver, 1983; Charalambous, Delaney, Hsu, & Mesa, 2010 ; Panaoura, Gagatsis, Deliyianni, & Elia, 2009). By understanding the full range of interpretations and representations of fractions, we are able to provide a coherent conceptual framework that underpins the ELE and its associated tasks.

2.2.2 Representations of fractions

In addition to fraction interpretations, there are also a number of ways that fractions can be represented. Graphical representations of fractions, such as area models (e.g., fraction circles, Geoboards) and linear models (e.g., fraction strips, Cuisenaire rods, number lines) are used extensively in fractions instruction and several studies demonstrate the promise of providing instruction that links these representations of fractions to the underlying fractions concepts (Kong, 2008; Paik, 2005; Pitta-Pantazi, Gray, & Christou, 2004; Yang & Reys, 2001) and support in actively making connections among the representations

(Ainsworth, 1999; Tabachneck, Leonardo, & Simon, 1994). The three most commonly cited include representations of area/region, number line and sets of objects (Duval, 2006). Another representation is modelled by liquid measures. Although research into the use of liquid measures is scarce, it has shown potential particularly to conceptual knowledge (Silver, 1983).

Table 2. Representations of fractions.

Representation	**Commentary**
Area, region	The area representation is the most common in textbooks and other instructional materials around the world (Alajmi, 2012; Pantziara & Philippou, 2012). It often uses a figure such as a circle or rectangle with a fractional part shaded. In context-based problems, items such as pies, pizzas and cakes are frequently used.
Number line	The number line representation involves students placing fractions on a number line or identifying the fraction that is shown on a number line. This requires the student to associate a point on the number line with the fraction a/b where each unit segment has been separated into *bc* equivalent line segments.
Sets of objects	Objects are grouped and used to represent whole sets. In the UK, the fractional part is often identified as a different colour or as a subset of the whole. Objects can be as varied as toys, circles, sweets, sticks or children.
Liquid measures	Reference is made to liquid measures in elementary educational materials from the USA (CCSSI, 2010; NCTM, 2000) and in the UK it is commonplace to find young children sharing objects, using everyday language to talk about capacity and solving problems (DfE, 2013; STA, 2012). It is uncommon to consider fractions in this way in Germany, one of the countries where the ELE is being tested.
Symbolic	The symbolic representation of a fraction uses the notation: numerator denominator Additionally, the numerator and denominator take on different meanings according to the type of representation. For example, as an area the denominator represents the number of parts the whole has been cut into and the numerator is the number of parts taken (Mamede, Nunes, & Bryant, 2005), whereas the numerator is compared with the denominator in the ratio model. Furthermore, in the quotient representation the numerator is the number of items to be shared and the denominator is how many people the item(s) must be shared by. The line represents a division.

In order to know with which representation German students are more familiar, RUB conducted a small analysis of national mathematics textbooks. The results of this analysis showed that for adding and subtracting fractions German students are most familiar with circles and less with sets of objects. IOE's visits to English schools showed that, similarly, English students are most familiar with circles and as a result tend to use circles as a first resort to represent a given fraction. England's National Numeracy Strategy (DfEE, 1999) and subsequent Primary National Strategy (DfES, 2006) encouraged teachers to use a range of representations including area/region representations beyond circles, sets of objects and number lines. Liquid measures are less commonly used.

These German and English findings concur with analyses of text books around the world by others (Alajmi, 2012; Pantziara & Philippou, 2012), that consistently show a skewed and limited exposure to a range of interpretations and representations. An over-reliance on any one representation limits students' conceptual understanding of fractions: for instance, area/region representations cannot exceed the number of partitions, which might impede children's learning of improper fractions (Smith, 2002). Indeed, each representation brings its own limitations. The number line, for example, "comprises a difficult model for students to manipulate" (Charalambous & Pitta-Pantazi, 2007) and placing fractions on the number line requires children to know that for the same denominator, the larger the numerator the larger the fraction and for the same numerator, the larger the denominator the smaller the fraction (Nunes, 2004).

In light of these findings about representations, to ensure robust learning we have taken the decision in iTalk2Learn to enable students access to a range of interpretations and representations within the learning platform because it is well-documented that use of multiple representations improves conceptual learning (Lamon, 1999; Rau, Aleven, & Rummel, 2013), as the constraints of each representation may be mitigated by exposure to others. This decision has implications for the design of the learning platform, for which structured tasks are included, and for the design of the ELE and the associated exploratory tasks. Using technology in iTalk2Learn enables students to perform actions on representations in a way that is difficult if not impossible away from the computer to enhance their learning. For example, a student is able to establish a relation between a part and a whole where it is possible to dynamically pull apart a partitioned whole while leaving the original whole intact, something that was not previously possible without destroying the original (Olive & Lobato, 2007). The design of the ELE has been completed taking aspects such as this into consideration and was reported upon in D3.2.

3. Relevant Intelligent Tutoring Systems

This section reviews relevant intelligent tutoring systems focusing mostly on the support provided within the two ITSs that we will utilise in the project (Whizz and Fractions Tutor). To cover the state-of-the-art we reviewed the literature for recent (last 5 years) ITSs but also used internet searches and our background knowledge and experience to identify work that is not published or necessarily peer reviewed. We were mainly interested in two strands: 1) reviewing how different systems approach sequencing of material since this relates to the recommendation aspect of the project (mainly WP2) and 2) the distinction between procedural and conceptual knowledge as this informs the tasks and possible interventions developed in WP1. With respect to the speech output and recognition strand of the project, D3.2 provided a detailed analysis but we also looked at these aspects in the rest of the systems that we reviewed.

The next section summarises our findings of the ITSs that were most relevant to our project in a table. We can see that most ITSs focus mainly on procedural knowledge because of their structured nature. Most of the systems in the area aim at adapting task selection and feedback to the individual learners' cognitive skills by employing models about the skills involved in solving the problems offered by the tutoring systems or models about the learners themselves. As such, they do not try to explicitly interleave in some adaptive manner structured and exploratory tasks. In addition (apart from the systems covered in D3.2), very few systems include natural language interaction. Some research efforts encourage students to provide text input and as expected some systems provide task descriptions and other instructions using speech synthesis. With respect to our domain we see that there is a wide coverage, but since this is mainly focused on procedural knowledge, there is ground for putting the emphasis of the iTalk2Learn learning goals on the development of conceptual knowledge in combination with procedural knowledge to foster robust learning.

Section 3.2 provides a detailed exposition of the two main tools in iTalk2Learn. In Section 5 we draw some related findings across the deliverable.

3.1 Relevant ITSs for mathematics

Table 3. Relevant ITSs for mathematics and their main features applicable to the iTalk2Learn project

ITS (or similar)	Relevant content to iTalk2Learn	Voice or Language interaction	Support: Providing feedback and/ or hints	Structured and/or Exploratory tasks	Adaptive Task Selection	Other relevant aspects
Active Math	"Rechnen mit Brüchen: Addition und Subtraktion" Adding & Subtracting fractions	LeAM project introduced tutorial dialogues to some structured problems	Immediate feedback, hints, worked examples	Mostly structured tasks. Some exploratory applets.	Yes	Open Student Model
RM Reasoning Mind	Fractions and Mixed Numbers (http://www.reasoningmind.org/syllabus/syllabus.php?n=2)		Feedback	Structured questions	Problem selection within a topic	Tasks just one part of a richer learning environment
Scoyo	Fractions Addition and Subtraction	Partly spoken feedback, spoken worked examples (2 avatars)	Feedback with immediate hint (2 levels)	Structured questions	Fixed Change between structured practice and reflection tasks (e.g. what do we do for adding two fraction)	Covers more than just mathematics

System	Relevant content to iTalk2Learn	Voice or Language Interaction	Support: Providing feedback and/ or hints	Structured and Exploratory tasks	Adaptive Task Selection	Other relevant aspects
Animal Watch	Fractions (http://animalwatch.arizona.edu/animalwatch_learning_objectives)	Partly spoken feedback, spoken worked examples (2 avatars)	Multimedia tutorial resources (from text explanations to worked examples and interactive solutions).	Problem solving in context of endangered species and tracing of their life. Includes some interactive games	Problem selection and difficulty according to skills observed	Basic problem solving skills development for pre- algebra topics. Motivation through understanding how mathematics could be applied in the real world (Cohen, Beal, & Adams, 2008)
Cogni-tive Tutors (in general)	Fractions Tutor and Math Tutor include relevant content (see Fractions Tutor below)	Some version provide spoken task description	Uses 'model tracing' (see also D2.1) to monitor progress and provide immediate hints and solution when needed (Anderson, et al., 1990)	Structured questions, broken down into steps	Mostly predefined sequence	See more details in Sections 3.2.2 and 3.2.3
Ms. Lind-quist	Focus on: writing expressions for algebraic word problems	Typed natural language interaction.	Concrete Example, Explanation, Worked Example, Decomposition Substitution (Heffernan, 2003)	Word problems to algebra expressions	Mostly predefined sequence	Some transcripts available at http://www.cs.cmu.edu/~neil/examplePage.html

3.1.1 Maths-Whizz

The Maths-Whizz proprietary online tutoring system is developed by Whizz Education. Since 2004 the company has been engaging in online, adaptive mathematics products which are used by thousands of students worldwide.

Maths-Whizz measures student achievement by continuously assessing individual student's levels in the different topics in mathematics and then identifying an overall "Maths Age" for each student. This is useful when used as a relative measure to compare profiles and abilities of students within a class in an easy and digestible manner. This empowers teachers to assess areas of attention for their class as a whole, as well as for individual students. This capability is being expanded to provide Ministries of Education with an ability to compare different regions or schools in terms of both mathematics ability and progress rate, raising the potential of evaluating the impact on mathematics of their various educational initiatives. Between 50,000 and 100,000 students are virtually tutored by Maths-Whizz in multiple countries at any one time, providing a significant amount of data for useful analysis and value added conclusions.

Maths-Whizz provides students with guided instruction and immediate feedback, which are both used as motivational tools. The system sets students up to answer questions by first presenting a short introduction, which explains how to complete the exercise successfully. As students work through the questions, they receive feedback according to their answers. When an incorrect answer is entered, Maths-Whizz analyses the answer to provide feedback appropriate to the error, encouraging students to elaborate and reflect about problem-solving strategies on their own before receiving the correct answer (see Fig. 1 for an example). Correct answers are rewarded with a celebratory response. Maths-Whizz selects the subsequent set of questions based upon a combination of a threshold for the exercise score and a post-exercise test score.[1]

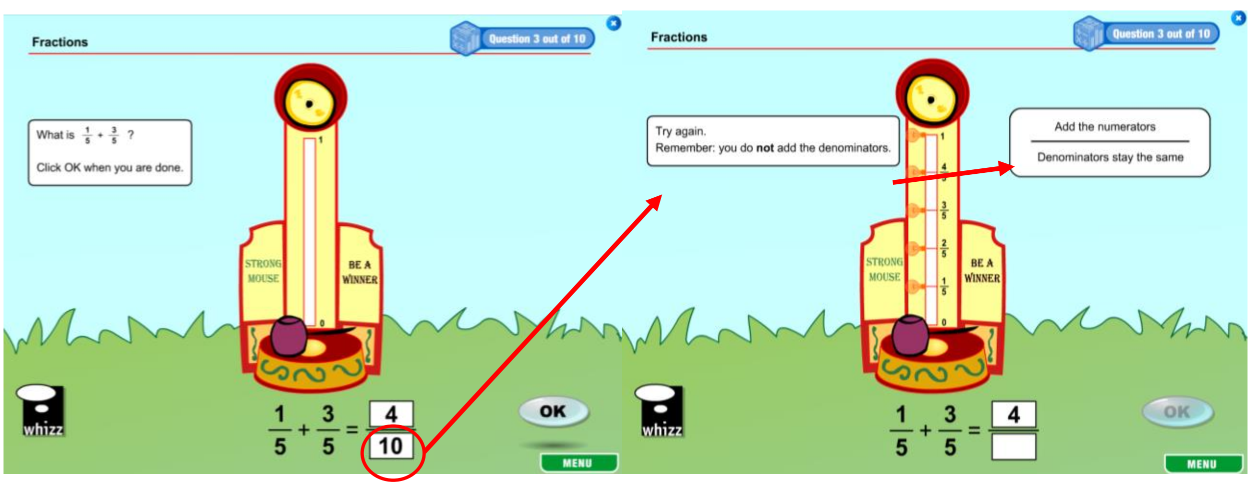

Figure 1. A Whizz question. A student's response where like denominators have been incorrectly added together results in feedback that states, "Remember: you do not add the denominators" and is followed up with "Add the numerators; Denominators stay the same".

[1] As the Maths-Whizz Tutor is a commercial product, detailed information with respect to their recommendation algorithm is not disclosed in this document. For the purposes of the work of the consortium, Whizz has provided the necessary information to the rest of the partners in the context of the non-disclosure agreement signed as part of the consortium agreement.

Maths-Whizz exercises use a range of graphical representations such as circles, rectangles, number lines, liquid measures and symbols within contexts that the students may be familiar. For English students, tasks are linked to the Mathematics National Curriculum of England and associated guidance (such as The National Numeracy Strategy and the National Primary Framework) that schools follow (giving a total of 44 objectives with associated exercises with normally 10 questions per exercise). For the purposes of the iTalk2Learn project we will make an effort to cover thoroughly all the fractions interpretations and representations, as opposed to following any one country's curriculum. There may therefore be a need to develop further exercises within Maths-Whizz to ensure a full range of tasks for the scope of the project.

3.1.2 Fractions Tutor

The Fractions Tutor is a web-based Cognitive Tutor for fractions. It was developed at Carnegie Mellon University and follows a long line of Cognitive Tutor development. The Cognitive Tutor technology for supporting student learning by doing is based on over 20 years of research (e.g. Aleven, McLaren & Sewall, 2009; Anderson, Corbett, Koedinger & Pelletier, 1995; Koedinger & Corbett, 2006). During these two decades, Cognitive Tutors have proven their effectiveness in several studies and across several domains (Van Lehn, 2011). Like other Cognitive Tutors, the Fractions Tutor was constructed in an interdisciplinary research team unifying experts from the field of Human-Computer-Interaction, Math Education and Psychology and was iteratively improved.

The Fractions Tutor (Rau, Aleven, & Rummel, 2009; Rau, et al., 2013; Rau, Aleven, Rummel, & Rohrbach, 2012) scaffolds the problem-solving process by enabling students to solve fraction problems step-by-step. As students solve problems they receive just-in-time feedback, which indicates whether they solved the problem-solving step correctly or incorrectly. Additionally, students have the possibility to ask for hints. Fractions Tutor hints have three different levels. The first level hint gives very abstract and vague information towards the solution, which should prompt students to elaborate on possible solution strategies on their own. The second level hint is a bit more specific concerning the solution strategy, but still leaves space for interpretation. Finally, the third level hint (the so-called 'bottom-out' hint), contains the solution of the respective problem-solving step. Implementation of three different levels of hints within the Fractions Tutor aims to force students to elaborate and reflect about problem-solving (strategies) on their own before receiving the correct answer.

Thematically speaking, the Fractions Tutor covers 10 main topics with nearly each topic including 18 problems. The total of 180 problems enable students to interact with the tutor for around 10 hours. The range of topic reaches from naming fractions (Unit 1), representing fractions (Unit 2), understanding proper fractions (Unit 3), understanding improper fractions (Units 4 and 5) over understanding the equivalence of fractions and making fractions equivalent (Units 6 and 7), comparing fractions with benchmarks (Unit 8) to adding and subtracting fractions (Units 9 and 10). Given that the iTalk2Learn project aims to foster robust learning in the domain of fractions (addition and subtraction), Units 9 and 10 of the Fractions Tutor are particularly interesting. Within these two units, four different types of problems are covered:

The first problem type (see Figure 2 below) focuses on adding fractions with like denominators. Within this first type of problem, students are challenged with finding the numeric expression of the fraction displayed either on a number line, circle or rectangle and to add the two fractions. Subsequently, students receive reflection prompts asking them, for example, to refer to the unit of the used fractions.

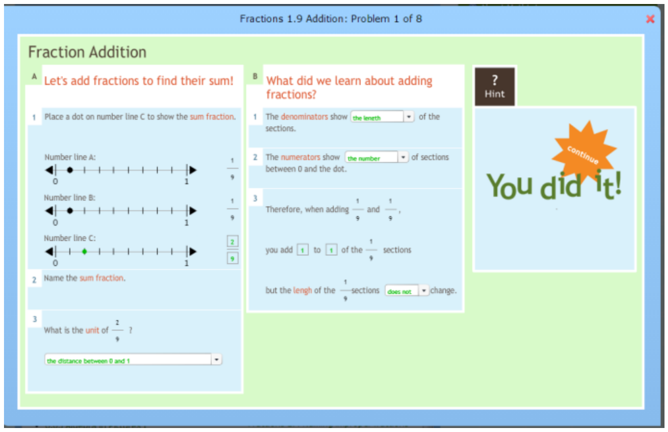

Figure 2. Adding like denominators n Fraction Tutor

The second problem type is structurally equal to the first one; the only difference lies in adding fractions with unlike denominators instead of adding fractions with like denominators (see Figure 3 below). Adding fractions with unlike denominators affords students to prepare the fractions for addition – they are required to make equivalent fractions through finding, for example, the least common denominator. Students can do so by manipulating (partitioning) the graphical representations.

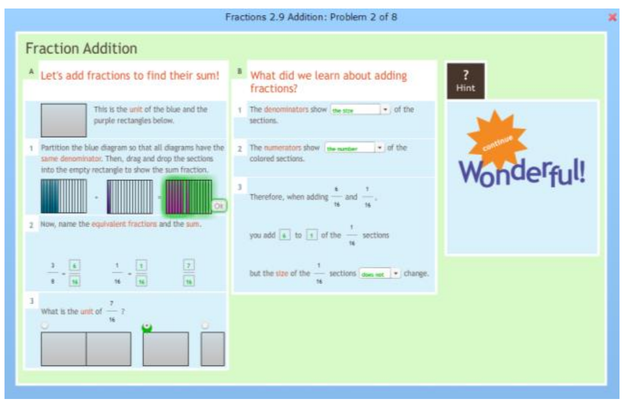

Figure 3. Adding unlike denominators in Fraction Tutor, making an equivalent fraction to make the same denominator before adding them two fractions.

The third problem group also refers to adding fractions with unlike denominators , but uses a combination of a worked example of the problem-solving procedure and a problem for practice. For the worked example and the problem, two different graphical representations are used, namely circles (worked example) and number lines (problem). In doing so, students should pay attention to make links between the two graphical representations expressing the same concept/ procedure.

The fourth problem group asks students to sort the sums displayed in different representations of fractions on the left side of the interface to the respective addition exercise on the right side of the interface. As was the case for the third problem group, this problem type also aims at facilitating students linking between different representations.

The commonality across the four problem types of the Fractions Tutor lies in the sequentiality of problem-solving.

3.1.3 Shared ITS characteristics of Whizz and Fractions Tutor

Whizz and Fractions Tutor's designs that will ensure their integration into the learning platform to enable students' robust fractions learning. Both ITSs require students to solve problems on a step-by-step basis and immediate feedback is provided to students to support procedural knowledge.

Both ITSs use different graphical representations such as circles, rectangles and number lines. The use of multiple graphical representations aims at fostering deep learning and is theoretically rooted in the Rational Number Project (see e.g. Behr, Harel, Post, & Lesh, 1992); (Post, Cramer, Behr, Lesh, & Harel, 1993).

Although students are able to interact with multiple graphical representations (by selecting answers, dragging and dropping, and - in the case of Fractions Tutor - further partitioning) in both ITSs, these interactions are limited in comparison to students' interactions within most ELEs (as reviewed in D3.2 and discussed in more detail in Section 4). Within ELEs, students can choose with which and with how many different graphical representations they would like to undertake a task. Additionally, particularly the ELE being developed in iTalk2Learn is designed to enable students to observe interdependencies across different representations, because students can see how changes in one type of representations lead to changes in the other type of representation. In contrast, the two ITSs do not allow students to select between different graphical representations, nor do they enable students to make explicit links across different representations. Students' interactions with representations within the ITSs only implicitly show that a fraction can be expressed through a circle as well as through a rectangle (and so on). Therefore, the ITSs' use of multiple graphical representations follows a purpose other than that of ELEs' multiple graphical representations, namely to support students by solving routine problems/ by performing structured activities.

In addition, both ITSs contain a significant number of exercises that can be drawn upon for the switching between structured and exploratory tasks as required. Despite the amount of pre-existing content (and in addition to translating Fraction Tutor content into German), it is likely that some further exercises will need to be designed in order to fulfill the requirements of iTalk2Learn's learning platform currently in design, particularly in relation to ensuring appropriate student access to a range of representations and interpretations.

4. Exploratory Learning Environments and Mathematics

We reported details regarding exploratory learning environments (ELEs) in general in D3.2 (see Section 2.1) and presented an extensive review of the relevant elements of ELEs. It is not our intention to repeat this here. However, in the context of WP1 it has been useful to revisit the partners' understanding with respect to exploratory learning environments not so much in relation to their interactive features, but their epistemic affordances and the claims designers and educators make with respect to mathematics learning.

The potential of ELEs lies in dealing with a key difficulty in mathematics learning; in that procedural knowledge (especially in elementary mathematics) often results in students learning rules that tend to dominate over conceptual knowledge. From the point of view of the learner, this sometimes results in very little understanding behind the rules, how and why the rules work and more importantly why they may want to learn them. As a solution to this challenge, researchers in the field of mathematics education investigate the use and design of appropriate exploratory environments (often referred to as 'microworlds'), which aim at fostering conceptual knowledge. The main basis for this work has mostly emerged from the educational approach of constructionism (or 'framework of action' as described in (DiSessa & Cobb, 2004) that puts exploratory learning in its core but also emphasizes that children learn most effectively by not only exploring their world but also being active by making (and sharing) artifacts. In the context of mathematics and microworlds or digital exploratory environments, this is particularly the case by allowing students to inspect, manipulate and investigate the structure of objects within the environment, construct and deconstruct their own and crucially, explore the mathematical relationships between and within the objects and the concepts and representations that make them accessible (Hoyles, 1993; Thompson, 1987).

Therefore, with respect to ELEs therefore in particular, the iTalk2Learn project will build upon a tradition of software design that has asserted the effectiveness of open, exploratory and expressive tools, particularly as a means of helping students develop conceptual knowledge: see, for examples, (de Jong & van Joolingen, 1998; Healy & Kynigos, 2010; Hoyles & Lagrange, 2010; Noss & Hoyles, 1996b).

However, as we elaborate in sub-section 4.1, research has also recognized the difficulty of finding a balance between allowing students to explore the environment, and steering them towards concepts that matter from a didactical point of view. This apparent lack of instructional support is often the criticism of the constructionist-oriented approaches (Kirschner, Sweller, & Clark, 2006; Mayer, 2004) including exploratory learning. This problem is exacerbated when, for example, due to the large number of students in classrooms the much-needed individual support is not feasible. This is precisely the challenge of iTalk2Learn. It may be relatively easy nowadays to develop a system that provides step by step guidance in structured tasks that tell students what to do and how, setting the students up to achieve the learning goals of the task without necessarily developing any deeper conceptual knowledge. Guiding students towards effective interaction without compromising the exploratory nature of the ELE is not so straightforward. Sub-section 4.2 elaborates on this.

4.1 Relevant ELEs for mathematics

Table 4 below summarises the most prominent ELEs that we have reviewed for the purposes of iTalk2Learn. We focus mostly on the examples we analysed thoroughly in the context of D3.2 and expand with three related systems. In combination with an older review (Olive & Lobato, 2007) this analysis provides further insights on how ELEs are designed to support students to construct mathematical knowledge and for fractions in particular.

We see that most exploratory environments are designed in an attempt to respond to the difficulties that learners face in associating procedural and conceptual knowledge and provide opportunities for making abstract concepts concrete and manipulable, exploiting the affordances of digital environments for dynamic and multiple representations.

With respect to fractions in particular, the review of the related ELEs and the associated literature indicates that the emphasis is in helping students make links between fractions symbols and pictorial representations. This analysis is influencing our design assumptions, drivers and conjectures as reported previously in D3.2. Other design decisions will be reported through task T3.2 and its corresponding deliverables D3.4.1 and D3.4.2. At this stage, the analysis has helped the consortium have a common understanding of what constitutes an ELE and where the potential of ELEs lies. As also observed in (Olive & Lobato, 2007), apart from the interactive nature of ELEs, their differences with more structured (question-answer) environments are firstly that they allow students to perform actions that are not otherwise necessarily possible or easy to do (e.g. on paper-pencil tasks), and secondly that they attempt to match interaction with mental operations that encourage conceptual (rather than just procedural) learning.

Another relevant aspect that we are taking into careful consideration is that ELEs in general have an intrinsic playful characteristic (some, such as games in particular, more than others). Our literature review and experience suggests that students enjoy interacting with them. This is not only an important motivational element but, as mentioned above, it also introduces a challenge, which (Noss & Hoyles, 1996a) refer to as the "play paradox". If a teacher introduces a new idea from outside the students' activity they are no longer playing, yet if it is not imported then the student may never encounter the idea. Therefore it requires effort on behalf of a teacher to turn students' play into meaningful mathematical activity through carefully-designed tasks (c.f. Biddlecomb & Whitmire, 1992). (Ainley, Pratt, & Hansen, 2006) introduce two constructs to inform the design of tasks: purpose and utility. Purpose is meaningful for the student, often a product or a solution to a problem; it is not the target knowledge a teacher may have set as the task objective. Utility is the construction of meaning for how, when and why the mathematical ideas are useful. Tasks designed with both purpose and utility address the play paradox.

The field has realized that, both from an interaction and learning perspective, it is essential to design an ELE in a way that can be used in ways that support students' learning but even in that case, without external support, this does not necessarily lead to learning (see McGrenere & Ho, 2000; Noss & Hoyles, 1996a). In iTalk2Learn the onus falls on Task 1.2 for the design of the appropriate tasks, Task 1.3 to interleave structured with exploratory activities and lastly WP2 work with respect to intelligent support for exploratory tasks. We elaborate on the latter in the next section.

Finally, we are considering the potential that ELEs have to enable reflection (Kong & Kwok, 2003) and reification of learning (Klopfer, Osterweil, & Salen, 2009). In their review of ELEs that support the learning of rational number concepts, Olive & Lobato (2007) identify that reflection was undertaken by students and carefully planned in by the teacher/researchers in every case. In iTalk2Learn, as we discussed in D3.2, the learning platform can create the conditions where we are able to support students to reflect on their learning through speech, in all its forms. For example, ‘inner speech’ (Vygotsky, 1978), dialogue with oneself, can facilitate self-regulation of one's behaviour and self-directed speech assists students to plan and coordinate their thoughts, actions and cognitive development (Zakin, 2007). Feedback responding to what students say can be provided through the system (see Section 3.3 of D3.2) to encourage students to reflect on their actions. Further reflection is also encouraged by students being required to act at a metacognitive level as they record what they did to share with peers or their teacher at a later date.

Table 4. Relevant (relatively recent) ELEs and key aspects of their design with respect to conceptual learning.

ELE (or similar)	Content	Designers expectations of students' interaction	How conceptual learning is achieved
Refraction	Halving and doubling fractions (http://centerforgamescience.org/portfolio/refraction/)	Game designed to provide rationale and motivation for fraction operations through bending and splitting lasers. Trial and error as a problem solving strategy.	Relational understanding with respect to fractions. Understanding of the concept of 'unit'
Quads	Geometric definitions (Hansen, 2008)	Initial trial and error leading to strategic game play.	Designed to help students focus on producing a definition by focusing on properties and relationships, rather than thinking about prototypical instantiations of a shape.
eXpresser	Algebra (M Mavrikis, Noss, E, & Hoyles, 2013)	Pattern creation as a means of focusing on mathematical structure and algebraic relationships (rather than 'pattern spotting')	Identifying relationships between animating objects supports an understanding of the concept of variables. Developing rules that count the number of tiles for animated patters provides rationale of algebra.
Logotron Visual Fractions (LVF)	Fractions (Lehotska & Kalas, 2005)	Teacher-led interaction with fraction representations. Activity authoring and student exploration and investigation of fraction properties and relations.	Links between different representations can be made to explore fractional properties and relationships
Gizmos	Making fractions, adding and subtracting fractions, converting improper fractions to mixed numbers (http://www.explorelearning.com)	Representations constructed by designers Manipulation of the virtual representation	Highlight one representation with specific tasks and goals to be accomplished per gizmo.
Graphical partitioning model	Allows learners to associate fraction symbols with corresponding graphic representations (Kong, Kwok, 2003)	Microworld interaction for making meaning about unit, part-size, part-whole and equivalence.	The GPM is designed to support three key stages of operations: perceiving the need of finding a common fractional part, finding a common fractional part and adding/subtracting of the fractional parts.
Conceptua Math	Grade 3 – 5 math including fractions (http://www.conceptuamath.com/)	Visual representations encourage students to build flexibility in their thinking through an instructional model that employs tasks with interactive representations.	Teacher-led lessons from concrete representations to numeric procedures. Story problems and 'hand-crafted' sequenced from concrete to abstract learning.

4.2 Intelligent support for ELE

We mentioned above that the richness of ELEs comes at the cost of the necessary pedagogical support, which (until recently) was assumed to be provided by teachers. This often results in ELEs being used as teacher demonstration tools or in one-to-one tutoring. The reason is that integrating an ELE in the classroom is a far from trivial process. A substantial body of research demonstrates that in order to ensure that students' interactions are effective in terms of learning, there is a need for significant pedagogic support. Recent work in the field (c.f. D2.1 for more details) indicates that it is possible to delegate some of the support to an intelligent system. While D2.1 provides more details from a technical perspective, below we summarize the state-of-the-art based on the project's challenge to design effective pedagogic support that guides students towards beneficial interactions without compromising their exploratory potential. We rely on previous analyses by members of the consortium (IOE & BBK), which resulted in a framework of pedagogical strategies for supporting student's exploration in ELEs. These are presented in Table 5 and are based on a collection of empirical evidence and theoretical perspectives (for more details see (M. Mavrikis, Gutierrez-Santos, Geraniou, & Noss, 2012) and Section 2.3.2 of D3.2 that shows an example of how intelligent support is provided).

Table 5. A framework of pedagogic strategies for student support in ELE

1. Supporting interaction with the learning environment
Introducing or reminding students of the environment's affordances
Helping students organise their working space
2. Encouraging goal-orientation
Structuring activities
Supporting students to set and work towards explicit goals
3. Exploiting learning opportunities
Directing students attention
Introducing cognitive conflicts, or counter-examples
Supporting reflection on actions and on task
(from Mavrikis et al., 2012)

The aforementioned framework acts as a guide for WP2 technical requirements and WP3 design requirements for the ELE under development. In particular, it highlights the need to design parts of the system with intelligent support in mind whilst taking into account pedagogical and HCI considerations. It is clear that the exploratory nature of the system and the freedom for students to interact with it should not be compromised by the need for intelligent support. However, as already recognised by a large body of work since the early 1990s (Orey & Nelson, 1990; Frye & Litman, 1995), there is a need for designing the system in a way that provides access to sufficient unambiguous information in order to enable inference based on students' interactions. The requirement that emerges is in making sure, in the ELE design, that misconceptions can be manifested and that there is sufficient amount and quality of the data (also referred to as 'bandwidth' by (Van Lehn, 1988) to the intelligent components. The challenge lies in making interaction explicit without being intrusive, interfering too much or limiting the exploratory nature of the ELE.

5. Summary and implications for iTalk2Learn

This deliverable presents a key building block in the iTalk2Learn project by preparing the ground for domain and pedagogical aspects that are required throughout the project. Against the background of different approaches to knowledge development, and based on this state-of-the-art review, several implications can be drawn for the iTalk2Learn project in general and for the design of the structured and exploratory tasks, and the intervention model in particular.

First, procedural knowledge (knowledge about and application of procedures) and conceptual knowledge (implicit or explicit understanding about underlying principles and structures of a domain) develop iteratively through different pedagogical approaches. We discuss in this section how these approaches can be reflected in ITSs and ELEs.

With view on the development of procedural knowledge, most ITSs afford students the practice of structured activities. The inherent structure of the questions guides students through the problem-solving process step-by-step and provides students with immediate feedback. In doing so, procedural knowledge is facilitated by reinforcing the steps required to find solutions using a drill-and-practice approach. With regard to the development of conceptual knowledge, ELEs afford students to discover the relevant/respective concepts about fractions. Within ELEs, the student drives the system (as opposed to the system driving the student through the problem-solving process), to help him/her develop understanding about fractions. While undertaking tasks in the ELE students make conjectures and investigations, ask questions, construct explanations, test their explanations (Dogan-Dunlop, 2003). In doing so, the ELE enables students to use different graphical representations of fractions (see Section 2.2). The possibility of using multiple graphical representations supports students in the development of representational flexibility, a core facet of conceptual knowledge (Cramer, Behr, T., & Lesh, 1997). Lastly, there is evidence to suggest that the possibilities for explicit reflection will contribute further to developing robust knowledge.

In the context of the iTalk2Learn project, procedural and conceptual knowledge are developed through the structured and exploratory tasks which are embedded within the project's learning platform. Table 6 summarizes how ITSs support procedural understanding and ELEs supports conceptual understanding. In practice, we will use the Fractions Tutor in Germany and Maths-Whizz activities in the UK. The alignment of the two different ITS to the two countries lies in the testing of the platform in two application scenarios that differ, which we believe is a strength of the developed platform. Because neither system was designed using all the fraction interpretations and representations, adjustments may be needed to make them fit for our purpose. The Fractions Tutor is currently being translated from American English to be fit-for-purpose in German schools. From a technical point of view, using two different systems also demonstrates the generality of the approach in adaptively selecting structured activities from either the Fractions Tutor or Whizz and combining them with exploratory learning activities (see D5.1 for more details).

Table 6. A summary of how Intelligent Tutoring Systems support procedural understanding and Exploratory Learning Environments support conceptual understanding

Intelligent Tutoring Systems	***Exploratory Learning Environments***
• Procedural approach to learning	• Conceptual approach to learning
• Closed environment which is system-driven, based on the student's past performance	• Open-ended environment that is student-driven, based on the student's own choices
• Provides tasks that require drill and practice to reinforce procedures	• Provides tasks that support students to discover and understand underlying concepts
• Scaffolds through increasingly difficult tasks and structured feedback	• Scaffolds through representations, tools and integrated feedback designed into the system
• Feedback instructs the student how to accurately complete the given task using the desired procedure	• Feedback is often as a result of the student's own actions. Other feedback makes suggestions as to what the student may wish to focus upon next
• The underlying structure is objective-driven focusing on what the student needs to do	• The underlying structure is conceptually-based focusing on what the student needs to understand

Finally, the review established that both structured and exploratory tasks require explicit pedagogical support, for which we are designing in the project. In doing so, we are considering the usual so-called 'assistance dilemma' within computer-supported learning environments. The assistance dilemma (Koedinger & Aleven, 2007) refers to the on-going pedagogical question regarding when should students learn with high or low (no) assistance in the form of instructions. Some argue for more assistance, as is the case for learning with ITS that provide hints and correct student's errors in every step of the process. In ELEs the assistance dilemma is even more important and transforms to a dilemma about when to interrupt students work. The requirement of support during the tasks in the ELE raises implications with respect to its design. It needs to provide access to sufficient unambiguous information in order to enable the systems' intelligence to draw inferences based on students' interactions. But this access to information has to be designed in a way that is not intrusive but make full use of students' actions in the user interface neither interfering too much nor limiting the exploratory nature of the ELE.

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